GLOBAL APPROXIMATION THEOREMS FOR GENERAL GAMMA TYPE OPERATORS

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Dedicated to Prof. P. N. Agrawal

Abstract. In this paper, we obtained some global approximation results for general Gamma type operators.
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1. Introduction

For a measurable complex valued and locally bounded function defined on $[0, \infty)$, Lupas and Müller [12] defined and studied some approximation properties of linear positive operators $\{G_n\}$ defined by

$$G_n(f;x) = \int_0^\infty g_n(x,u) f\left(\frac{n}{u}\right) du,$$

where

$$g_n(x,u) = \frac{x^{n+1}}{n!} e^{-xu} u^n, \ x > 0.$$

In [13], Mazhar gives an important modifications of the Gamma operators using the same $g_n(x,u)$

$$\begin{aligned} F_n(f;x) &= \int_0^\infty \int_0^\infty g_n(x,u) g_{n-1}(u,t) f(t) du dt \\ &= \frac{(2n)! x^{n+1}}{n!(n-1)!} \int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}} f(t) dt, \ n > 1, \ x > 0. \end{aligned}$$

Recently, Karsli [7] considered the following Gamma type linear and positive operators

$$\begin{aligned} L_n(f;x) &= \int_0^\infty \int_0^\infty g_{n+2}(x,u) g_n(u,t) f(t) du dt \\ &= \frac{(2n+3)! x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t) dt, \ x > 0, \end{aligned}$$

and obtained some approximation results.
In [11], Karsli and Özarslan obtained some local and global approximation results for the operators $L_n(f;x)$.
In 2007, Mao [14] define the following generalised Gamma type linear and positive operators

$$\begin{aligned} M_{n,k}(f;x) &= \int_0^\infty \int_0^\infty g_n(x,u) g_{n-k}(u,t) f(t) du dt \\ &= \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t) dt, \ x > 0, \end{aligned}$$

which includes the operators $F_n(f;x)$ for $k=1$ and $L_{n-2}(f;x)$ for $k=2$.
Some approximation properties of $M_{n,k}$ were studied in [8] and [9]. Several authors obtain the global approximation results for different operators (see [1], [3] and [4]).
We can rewrite the operators $M_{n,k}(f;x)$ as

$$M_{n,k}(f;x) = \int_0^\infty K_{n,k}(x,t) f(t) dt, \tag{1.1}$$

where

$$K_{n,k}(x,t) = \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \frac{t^{n-k}}{(x+t)^{2n-k+2}}, \ \ x,t \in (0,\infty).$$

In this paper, we study some global approximation results of the operators $M_{n,k}$. Let $p \in N_0$(set of non-negative integers), $f \in C_p$, where C_p is a polynomial weighted space with the weight function w_p,

$$w_0(x) = 1, \ \ w_p(x) = \frac{1}{1+x^p}, \ \ p \geq 1, \tag{1.2}$$

and C_p is the set of all real valued functions f for which $w_p f$ is bounded and uniformly continuous on $[0,\infty)$.
The norm on C_p is defined by

$$||f||_p = \sup_{x\in[0,\infty)} w_p(x)|f(x)|, \ \ f \in C_p[0,\infty).$$

We also consider the following Lipschitz classes:

$$\omega_p^2(f;\delta) = \sup_{h\in(0,\delta]} ||\Delta_h^2 f||_p,$$

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x),$$

$$\omega_p^1(f;\delta) = \sup\{w_p(x)|f(t)-f(x)| : |t-x| \leq \delta \, and \, t,x \geq 0\},$$

$$Lip_p^2\alpha = \{f \in C_p[0,\infty) : \omega_p^2(f;\delta) = O(\delta^\alpha) \, as \, \delta \to 0^+\},$$

where $h > 0$ and $\alpha \in (0,2]$.

2. **Auxiliary Results**

In this section we give some preliminary results which will be used in the proofs of our main theorems.
Let us consider

$$e_m(t) = t^m,\ \ \varphi_{x,m}(t) = (t-x)^m,\ \ m \in N_0.$$

Lemma 1. [8] *For any $m \in N_0$(set of non-negative integers), $m \leq n-k$*

$$M_{n,k}(t^m;x) = \frac{[n-k+m]_m}{[n]_m}x^m \tag{2.1}$$

where $n, k \in N$ and $[x]_m = x(x-1)...(x-m+1), [x]_0 = 1, x \in R$.
In particular for m = 0, 1, 2... in (2.1) we get

(i) $M_{n,k}(1;x) = 1$,

(ii) $M_{n,k}(t;x) = \dfrac{n-k+1}{n}x$,

(iii) $M_{n,k}(t^2;x) = \dfrac{(n-k+2)(n-k+1)}{n(n-1)}x^2$.

Lemma 2. [8] *Let $m \in N_0$ and fixed $x \in (0,\infty)$, then*

$$M_{n,k}(\varphi_{x,m};x) = \left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{(n-m+j)!(n-k+m-j)!}{n!(n-k)!}\right)x^m.$$

Lemma 3. *For $m = 0,1,2,3,4$, one has*

(i) $M_{n,k}(\varphi_{x,0};x) = 1$,

(ii) $M_{n,k}(\varphi_{x,1};x) = \dfrac{1-k}{n}x$,

(iii) $M_{n,k}(\varphi_{x,2};x) = \dfrac{k^2-5k+2n+4}{n(n-1)}x^2$,

(iv) $M_{n,k}(\varphi_{x,3};x) = \dfrac{-k^3+12k^2-17k+n(18-12k)+24}{n(n-1)(n-2)}x^3$,

(v) $M_{n,k}(\varphi_{x,4};x) = \dfrac{k^4-22k^3+k^2(143+12n)-k(314+108n)+12n^2+268n+192}{n(n-1)(n-2)(n-3)}x^4$,

(vi) $M_{n,k}(\varphi_{x,m};x) = O\left(n^{-[(m+1)/2]}\right)$.

Proof. Using Lemma 2, we get Lemma 3. □

Theorem 1. *For the operators $M_{n,k}$ and for fixed $p \in N_0$, there exists a positive constant $N_{p,k}$ such that*

$$w_p(x)M_{n,k}\left(\frac{1}{w_p};x\right) \leq N_{p,k}. \tag{2.2}$$

Furthermore, for all $f \in C_p[0,\infty)$, we have

$$\|M_{n,k}(f;.)\|_p \leq N_{p,k}\|f\|_p, \tag{2.3}$$

which guarantees that $M_{n,k}$ maps $C_p[0,\infty)$ into $C_p[0,\infty)$.

Proof. For $p=0$, (2.2) follows immediately. Using Lemma 1, we get

$$\begin{aligned} w_p(x)M_{n,k}\left(\frac{1}{w_p};x\right) &= w_p(x)\left(M_{n,k}(e_0;x)+M_{n,k}(e_p;x)\right) \\ &= w_p(x)\left(1+\frac{(n-p)!(n-k+p)!}{n!(n-k)!}x^p\right) \\ &\leq N_{p,k}w_p(x)(1+x^p)=N_{p,k}, \end{aligned}$$

where

$$N_{p,k}=\max\left\{\sup_n \frac{(n-p)!(n-k+p)!}{n!(n-k)!},1\right\}.$$

Observe that for all $f\in C_p$ and every $x\in(0,\infty)$, we get

$$\begin{aligned} w_p(x)\left|M_{n,k}(f;x)\right| &\leq w_p(x)\frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}}|f(t)|\frac{w_p(t)}{w_p(t)}dt \\ &\leq \|f\|_p w_p(x)M_{n,k}\left(\frac{1}{w_p};x\right) \\ &\leq N_{p,k}\|f\|_p. \end{aligned}$$

Taking supremum over $x\in(0,\infty)$, we get (2.3). □

Lemma 4. *For the operators $M_{n,k}$ and fixed $p\in N_0$, there exists a positive constant $N_{p,k}$ such that*

$$w_p(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right)\leq N_{p,k}\frac{x^2}{n}.$$

Proof. Using Lemma (3), we can write

$$\begin{aligned} w_0(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_0(t)};x\right) &= \frac{k^2-5k+2n+4}{n(n-1)}x^2 \\ &\leq N_{p,k}\frac{x^2}{n}, \end{aligned}$$

which gives the result for $p=0$.
Let $p\geq 1$. Then using Lemma 1 and Lemma 3, we get

$$\begin{aligned} M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right) &= M_{n,k}(e_{p+2};x)-2xM_{n,k}(e_{p+1};x)+x^2M_{n,k}(e_p;x)+M_{n,k}(\varphi_{x,2};x) \\ &= \frac{(n-p-2)!(n-k+p+2)!}{n!(n-k)!}x^{p+2}-2\frac{(n-p-1)!(n-k+p+1)!}{n!(n-k)!}x^{p+2} \\ &+ \frac{(n-p)!(n-k+p)!}{n!(n-k)!}x^{p+2}+\frac{k^2-5k+2n+4}{n(n-1)}x^2 \\ &\leq N_{p,k}\frac{x^2}{n}(1+x^p), \end{aligned}$$

where $N_{p,k}$ is a positive constant. Hence, the proof is completed. □

3. **Rate of Convergence**

Let $p \in N_0$. By $C_p^2[0,\infty)$, we denote the space of all functions $f \in C_p[0,\infty)$ such that $f', f'' \in C_p[0,\infty)$.

Theorem 2. *Let $p \in N_0$, $n \in N$ and $g \in C_p^1[0,\infty)$, there exists a positive constant $N_{p,k}$ such that*

$$w_p(x)|M_{n,k}(f;x) - f(x)| \leq N_{p,k}\|f'\|_p \frac{x}{\sqrt{n}}$$

for all $x \in (0,\infty)$.

Proof. We have

$$f(t) - f(x) = \int_x^t f'(v)dv.$$

By using linearity of $M_{n,k}$ we get

$$M_{n,k}(f;x) - f(x) = M_{n,k}\left(\int_x^t f'(v)dv; x\right). \tag{3.1}$$

Remark that

$$\left|\int_x^t f'(v)dv\right| \leq \|f'\|_p \left|\int_x^t \frac{dv}{w_p(v)}\right| \leq \|f'\|_p |t-x| \left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right).$$

From (3.1) we obtain

$$w_p(x)|M_{n,k}(f;x) - f(x)| \leq \|f'\|_p \left\{M_{n,k}(|\varphi_{x,1}|;x) + w_p(x)M_{n,k}\left(\frac{|\varphi_{x,1}|}{w_p(t)};x\right)\right\}.$$

Using Cauchy-Schwarz inequality, we can write

$$M_{n,k}(|\varphi_{x,1}|;x) \leq \left(M_{n,k}(|\varphi_{x,2}|;x)\right)^{1/2},$$

$$M_{n,k}\left(\frac{|\varphi_{x,1}|}{w_p(t)};x\right) \leq \left(M_{n,k}\left(\frac{1}{w_p(t)};x\right)\right)^{1/2}\left(M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right)\right)^{1/2}.$$

Using Leema 3, Theorem 1 and Lemma 4, we obtain

$$w_p(x)|M_{n,k}(f;x) - f(x)| \leq N_{p,k}\|f'\|_p \frac{x}{\sqrt{n}}.$$

□

Lemma 5. *Let $p \in N_0$, If*

$$H_{n,k}(f;x) = M_{n,k}(f;x) - f\left(x + \frac{1-k}{n}x\right) + f(x), \tag{3.2}$$

then there exists a positive constant $N_{p,k}$ such that for all $x \in (0,\infty)$ and $n \in N$, we have

$$w_p(x)|H_{n,k}(g;x) - g(x)| \leq N_{p,k}||g''||_p \frac{x^2}{n}$$

for any function $g \in C_p^2$.

Proof. From Lemma 1, we observe that the operators $H_{n,k}$ are linear and reproduce the linear functions.
Hence

$$H_{n,k}(\varphi_{x,1};x)=0.$$

Let $g\in C_p^2$. By the Taylor formula one can write

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t (t-v)g''(v)dv,\ \ t\in(0,\infty).$$

Then,
$|H_{n,k}(g;x)-g(x)|$

$$\begin{aligned} &= |H_{n,k}(g(t)-g(x));x| = \left|H_{n,k}\left(\int_x^t (t-v)g''(v)dv;x\right)\right| \\ &= \left|M_{n,k}\left(\int_x^t (t-v)g''(v)dv;x\right)-\int_x^{x+\frac{1-k}{n}x}\left(x+\frac{1-k}{n}x-v\right)g''(v)dv\right|. \end{aligned}$$

Since

$$\left|\int_x^t (t-v)g''(v)dv\right| \le \frac{\parallel g''\parallel_p (t-x)^2}{2}\left(\frac{1}{w_p(x)}+\frac{1}{w_p(t)}\right)$$

and

$$\left|\int_x^{x+\frac{1-k}{n}x}\left(x+\frac{1-k}{n}x-v\right)g''(v)dv\right| \le \frac{||g''||_p}{2w_p(x)}\left(\frac{1-k}{n}x\right)^2,$$

we get

$$w_p(x)|H_{n,k}(g;x)-g(x)| \le \frac{\|g''\|_p}{2}\left[M_{n,k}(\varphi_{x,2};x)+w_p(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right)\right]+\frac{\|g''\|_p}{2}\left(\frac{1-k}{n}x\right)^2.$$

Hence by Lemma 4, we obtain

$$w_p(x)|H_{n,k}(g;x)-g(x)| \le N_{p,k}\|g''\|_p\frac{x^2}{n}$$

for any function $g\in C_p^2$. The Lemma is proved. □

The next theorem is the main result of this section.

Theorem 3. *Let $p\in N_0$, $n\in N$ and $f\in C_p[0,\infty)$, then there exists a positive constant $N_{p,k}$ such that*

$$w_p(x)\left|M_{n,k}(f;x)-f(x)\right| \le N_{p,k}\omega_p^2\left(f,\frac{x}{\sqrt{n}}\right)+\omega_p^1\left(f,\frac{1-k}{n}x\right).$$

Furthermore, if $f\in Lip_p^2\alpha$ for some $\alpha\in(0,2]$, then

$$w_p(x)\left|M_{n,k}(f;x)-f(x)\right| \le N_{p,k}\left(\frac{x^2}{n}\right)^{\alpha/2}+\omega_p^1\left(f,\frac{1-k}{n}x\right),$$

holds.

Proof. Let $p \in N_0$, $f \in C_p[0,\infty)$ and $x \in (0,\infty)$ be fixed. We consider the Steklov means of f by f_h and given by the formula

$$f_h(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\{2f(x+s+t) - f(x+2(s+t))\}dsdt,$$

for $h > 0$ and $x \geq 0$. We have

$$f(x) - f_h(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta^2_{s+t}f(x)dsdt,$$

which gives

$$\|f - f_h\|_p \leq \omega_p^2(f,h). \tag{3.3}$$

Furthermore, we have

$$f_h''(x) = \frac{1}{h^2}\left(8\Delta^2_{h/2}f(x) - \Delta^2_h f(x)\right),$$

and

$$\|f_h''\|_p \leq \frac{9}{h^2}\omega_p^2(f,h). \tag{3.4}$$

From (3.3) and (3.4) we conclude that $f_h \in C_p^2[0,\infty)$ if $f \in C_p[0,\infty)$.
Moreover

$$\begin{aligned} |M_{n,k}(f;x) - f(x)| &\leq H_{n,k}(|f(t) - f_h(t);x|) + |f(x) - f_h(x)| \\ &+ |H_{n,k}(f_h;x) - f_h(x)| + \left|f\left(x + \frac{1-k}{n}x\right) - f(x)\right|, \end{aligned}$$

where $H_{n,k}$ is defined in (3.2).
Since $f_h \in C_p^2[0,\infty)$ by the above, it follows from Theorem 1 and Lemma 5 that

$$\begin{aligned} w_p(x)\,|M_{n,k}(f;x) - f(x)| &\leq (N+1)\|f - f_h\|_p + N_{p,k}\|f_h''\|_p\frac{x^2}{n} \\ &+ w_p(x)\left|f\left(x + \frac{1-k}{n}x\right) - f(x)\right|. \end{aligned}$$

By (3.3) and (3.4), the last inequality yields that

$$w_p(x)\,|M_{n,k}(f;x) - f(x)| \leq N_{p,k}\omega_p^2(f;h)\left(1 + \frac{1}{h^2}\frac{x^2}{n}\right) + \omega_p^1\left(f, \frac{1-k}{n}x\right).$$

Thus, choosing $h = \frac{x}{\sqrt{n}}$, the first part of the proof is completed.
The remainder of the proof can be easily obtained from the definition of the space $Lip_p^2\alpha$. □

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References

[1] Becker M. Global approximation theorems for Szász-Mirakyan and Baskakov operators in polynomial weight spaces. Indiana University Mathematics Journal 1978; 27 (1): 127-142.
[2] DeVore RA, Lorentz GG. Constructive Approximation. Berlin: Springer, 1993.
[3] Felten M. Local and global approximation theorems for positive linear operators. J. Approx. Theory 1998; 94: 396-419.
[4] Finta Z. Direct local and global approximation theorems for some linear positive operators. Analysis in Theory and Applications 2004; 20 (4): 307-322.
[5] İzgi A. Voronovskaya type asymptotic approximation by modified gamma operators. Appl. Math. Comput. 2011; 217: 8061-8067.
[6] İzgi A, Büyükyazici I. Approximation and rate of approximation on unbounded intervals. Kastamonu Edu. J. Okt. 2003; 11: 451-460(in Turkish).
[7] Karsli H. Rate of convergence of a new Gamma type operators for the functions with derivatives of bounded variation. Math. Comput. Modell. 2007; 45 (5-6): 617-624.
[8] Karsli H. On convergence of general Gamma type operators. Anal. Theory Appl. 2011; Vol. 27, No.3: 288-300.
[9] Karsli H, Agrawal PN, Goyal M. General Gamma type operators based on q-integers. Appl. Math. Comput. 2015; 251: 564-575.
[10] Karsli H, Gupta V, Izgi A. Rate of pointwise convergence of a new kind of gamma operators for functions of bounded variation. Appl. Math. Letters 2009; 22: 505-510.
[11] Karsli H, Özarslan MA. Direct local and global approximation results for operators of gamma type. Hacet. J. Math. Stat. 2010; 39: 241-253.
[12] Lupas A, Müller M. Approximationseigenschaften der Gammaoperatren. Mathematische Zeitschrift 1967; 98: 208-226.
[13] Mazhar SM. Approximation by positive operators on infinite intervals. Math. Balkanica 1991; 5 (2): 99-104.
[14] Mao LC. Rate of convergence of Gamma type operator. J. Shangqiu Teachers Coll. 2007; 12: 49-52.
[15] Rempulska L, Tomczak K. Approximation by certain linear operators preserving x^2. Turkish J. Math. 2009; 33: 273-281.